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Comparative study and graphical representation of certain linear positive operators and their King-type modification

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Abstract

This paper provides a study on King-type modification of some linear positive operators, viz., Lupaş, Szasz-Mirakjan, and Baskakov operators. In the paper, density theorems for the operator and their modifications are presented and comparisons on the rate of convergence between original and modified operators are mentioned. In support to the mathematical results on convergence, some graphical presentation is shown to get a clear idea of better modification.

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1 Introduction

This paper aims to provide a comparative study between certain linear positive operators and their King-type modifications. In support to the mathematical results on convergence, some graphical presentation is shown, as in [10]. These modifications are derived with the aim of getting better approximations properties than the originally defined operators. Such results are inspired by the work that has been done by J. P. King for Bernstein operators.

To know the properties of linear positive operators, it is required to check their behavior at test functions, i.e., $\{1, x, x^2\}$. Some operators preserve the test functions $\{1, x\}$ but they fail to preserve x^2, instead, they give an expression that converges to x^2 as n tends to infinity.

King's modification for Bernstein operators was proposed with the intention of preserving the function x^2. That modification turned out to be better in convergence properties compared to the Bernstein operators. This idea inspires many researchers to construct different types of modifications of well-known operators by preserving functions like x, x^2, e^x, 2^{nx}, etc. In this paper, certain modifications are discussed along with the comparative study of the properties of those modifications with original operators.

2 Preliminaries

1. $\mathcal{C}[0, \infty)$ is the space of all continuous functions on $[0, \infty)$.
2. $\mathcal{C}_b[0, \infty)$ is the subspace of all continuous bounded functions.

Theorem 2.1. *Korovkin-type theorem*
Let $\{L_i\ ;\ i = 0, 1, 2, \ldots\}$ *be a sequence of monotone and linear operators from a set of continuous functions on* $[a, b]$*, i.e.,* $\mathcal{C}[a, b]$ *to itself. If the sequence* $\{L_i\ ;\ i = 0, 1, 2, \ldots\}$ *converges uniformly to* f *for the functions* $f(x) = x^k$,

$a \le x \le b$, where $k = 0,1$ or 2, then, the sequence $\{L_i\ ;\ i = 0,1,2,\ldots\}$ converges uniformly to f for all f in $\mathcal{C}[a,b]$.

Definition 1. *Moduli of continuity [2]*
For $f \in \mathcal{C}[0,c]$, the modulus of continuity of f, denoted by $w(f;r)$, is defined as

$$w(f;r) = \sup_{|a-b|\le r} \{|f(a) - f(b)|\},$$

for $a, b \in [0,c]$.

Properties of modulus of continuity

1. For $0 < r_1 < r_2$, $w(f,r_1) < w(f,r_2)$.
2. f is uniformly continuous if and only if $\lim_{\delta\to 0} w(f,\delta) = 0$.
3. For $a > 0$, $w(f,ar) < (1+a)w(f,r)$.

3 Lupaş operator and its King-type modification

This operator was proposed by A. Lupaş at the international meeting held in Germany [3] (1995) and briefly studied by O. Agratini [4]. It is defined as

$$\mathcal{L}_n(f,x) = (1-a)^{nx} \sum_{j=0}^{\infty} \frac{(nx)_j}{j!} a^j f\left(\frac{j}{n}\right), |a| < 1, x \ge 0, \tag{3.1}$$

with $f : \mathbb{R}^+ \to \mathbb{R}, n \in \mathbb{N}$,
where $(nx)_0 = 1$ and $(nx)_j = (nx)(nx+1)\cdots(nx+j-1),\ j \in \mathbb{N}$.

Agratini [4] found $a = \dfrac{1}{2}$ for this operator to satisfy $\mathcal{L}_n(e_1;x) = e_1(x)$ where $e_i(x) = x^i,\ i = 0,1,2$. For that value of a, Agratini defined the following operator that is derived from (3.1),

$$\mathcal{L}_n(f,x) = 2^{-nx} \sum_{j=0}^{\infty} \frac{(nx)_j}{2^j j!} f\left(\frac{j}{n}\right), \quad n \in \mathbb{N}. \tag{3.2}$$

It is known that for this given operator,
$\mathcal{L}_n(e_0;x) = e_0(x),\ \mathcal{L}_n(e_1;x) = e_1(x)$ and $\mathcal{L}_n(e_2;x) = e_2(x) + \dfrac{2e_1(x)}{n}$.

It is to be noted that this operator satisfies the Korovkin-type theorem 2.1. Now, for the comparative study of operators, it is required to discuss some quantitative estimates for the rate of convergence. We give estimates concerning the pointwise convergence in terms of the usual modulus of continuity of f, which is defined by (1).

Theorem 3.1. *[4] Let $\mathcal{L}_n$ be defined by* (3.2) *and* $b > 0$. *Then,*

$$|(\mathcal{L}_n f)(x) - f(x)| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{2b}{n}}\right) w(f;\delta),\ x \in [0,b], \tag{3.3}$$

Where $w(f;\delta)$ is defined in (1) and $\delta > 0$.

In this theorem, if δ is taken as $\frac{1}{\sqrt{n}}$, then the inequality becomes,

$$|(\mathcal{L}_n f)(x) - f(x)| \leq (1 + \sqrt{2b}) w\left(f;\frac{1}{\sqrt{n}}\right),\ x \in [0,b].$$

Here, as n tends to ∞, δ goes to 0 and if δ goes to 0, $w(f;\delta)$ goes to 0 as f is uniformly continuous on $[0,b]$. Therefore, this theorem proves that $\mathcal{L}_n f$ converges to f as $n \to \infty$.

3.1 King-type modification of Lupaş operators

This modification was given by F. Dirik in 2007 [5]. It is defined as

$$\mathcal{L}_n^*(f;x) = 2^{-nr_n(x)} \sum_{j=0}^{\infty} \frac{(nr_n(x))_j}{2^j j!} f\left(\frac{j}{n}\right), \quad n \in \mathbb{N}. \tag{3.4}$$

Where, $f \in C[0,\infty)$ and $r_n(x)$ is a sequence of continuous functions on $[0,\infty)$, which is defined as

$$r_n(x) = -\frac{1}{n} + \sqrt{x^2 + \frac{1}{n^2}}, \quad n \in \mathbb{N}. \tag{3.5}$$

Here, by taking $r_n(x) = x$, this operator becomes the original Lupaş operator mentioned in (3.2). It can be checked that,

$$\mathcal{L}_n^*(e_0;x) = e_0(x), \quad \mathcal{L}_n^*(e_1;x) = r_n(x) \quad \text{and} \quad \mathcal{L}_n^*(e_2;x) = (r_n(x))^2 + \frac{2r_n(x)}{n}.$$

For these values of the operator at test functions, Korovkin theorem for the operator $\mathcal{L}_n^*$ is given by,

Theorem 3.2. *[5] For the positive linear operator $\mathcal{L}_n^*$ defined as* (3.4) *and the sequence $\{r_n(x)\}$ defined as* (3.5)*,*

(i) $\mathcal{L}_n^*$ is a positive linear operators on $\mathcal{C}[0,\infty)$, $n \in \mathbb{N}$,

(ii) $\mathcal{L}_n^*(e_2;x) = e_2(x) = x^2, \quad n \in \mathbb{N}, x \in [0,\infty)$,

(iii) $\lim_{n\to\infty} \mathcal{L}_n^*(f;x) = f(x)$, on $[0,b]$.

Now for the rate of convergence, the following theorem is to be considered.

Theorem 3.3. *[5] Let $\mathcal{L}_n^*$ defined as* (3.4) *and the sequence $\{r_n(x)\}$ defined as* (3.5)*, then for $x \in [0,b]$ and any $\delta > 0$, we have*

$$|(\mathcal{L}_n^* f)(x) - f(x)| \leq w(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x - r_n(x))}\right]. \tag{3.6}$$

Now, to compare both the estimates of the rate of convergence for Lupaş and its King-type modifications, for every $x \geq 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} x^2 &\leq x^2 + \frac{1}{n^2} \\ i.e., \quad x &\leq \sqrt{x^2 + \frac{1}{n^2}} \end{aligned}$$

.
Therefore, it can be written as

$$2x\left(x + \frac{1}{n} - \sqrt{x^2 + \frac{1}{n^2}}\right) \leq \frac{2x}{n}$$

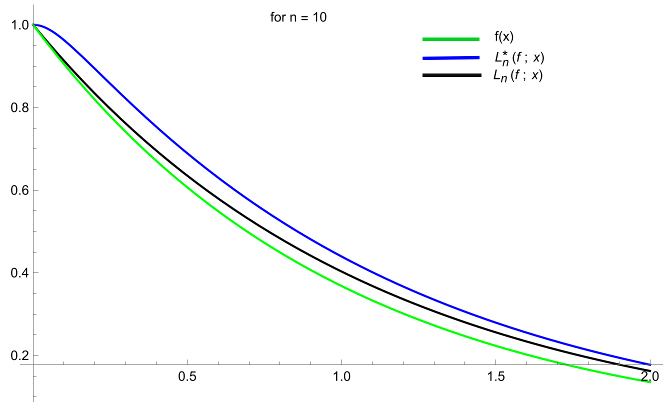
i.e.,

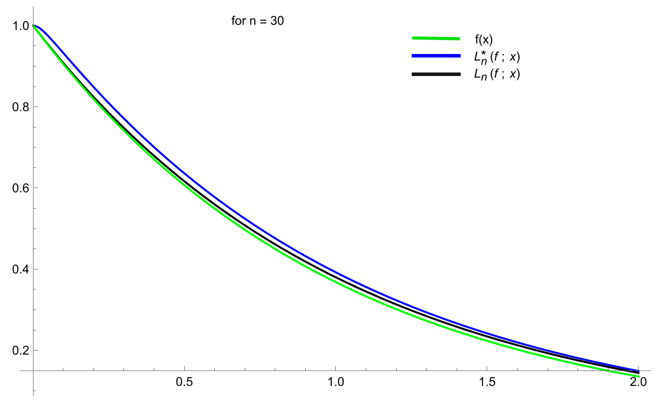
$$2x(x - r_n(x)) \leq \frac{2x}{n}.$$

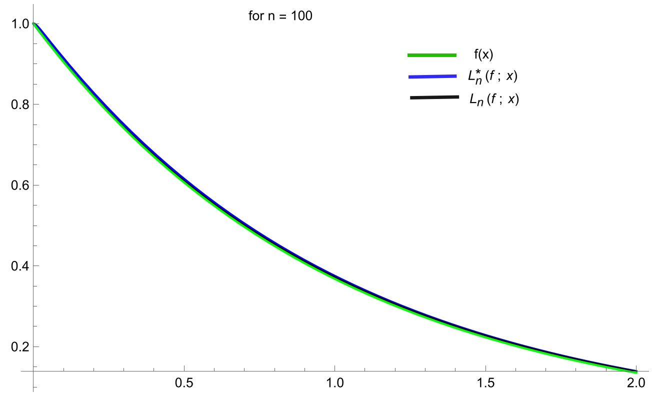
Here, the left side of this inequality is the rate of convergence for King-type modification, while the other one is for original Lupaş operators. It can be seen from this observation that the King-type modification of Lupaş gives better estimates than that of Lupaş operators.

3.2 Graphical representation for Lupaş and its King-type modifications

After summing up the theoretical estimates for the rate of convergence, we try to show the difference between those two types of the operator through graphical representation. For a particular function, a graph of the function, Lupaş operator at that function and King-type Lupaş operator at the same function are shown for different values of n. Here, the function is taken as $f(x) = e^{-x}$.







It can be seen that with increasing values of n, all the curves overlap. Therefore, as $n \to \infty$, the operator converges to given function.

4 Szász-Mirakjan operator and its King-type modification

The Szász-Mirakjan operators were proposed by Otto Szász and G. M. Mirakjan independently [6]. It is defined as

$$\mathcal{S}_n(f,x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f\left(\frac{j}{n}\right), \tag{4.1}$$

where $x \in [0,\infty)$ and $n \in \mathbb{N}$.
It can be checked that
$\mathcal{S}_n(e_0;x) = e_0(x),\ \mathcal{S}_n(e_1;x) = e_1(x)$ and $\mathcal{S}_n(e_2;x) = e_2(x) + \dfrac{e_1(x)}{n}$.
It is to be observed that $\mathcal{S}_n(f)$ converges to f for the test function. Therefore, by Korovkin-type theorem, $\mathcal{S}_n(f)$ converges to f for all continuous functions on $[0,b]$. Now, the rate of convergence of the Szász-Mirakjan operators is estimated by the following theorem.

Theorem 4.1. *[6] For every $f \in \mathcal{C}_b[0,\infty)$, $x \geq 0$ and $n \in \mathbb{N}$*

$$|\mathcal{S}_n(f,x) - f(x)| \leq 2w(f,\alpha_x), \tag{4.2}$$

where, $\alpha_x = \sqrt{\dfrac{x}{n}}$.

Here, it is to be noticed that as n goes to ∞, α_x goes to 0 and $w(f, \alpha_x) \to 0$. Therefore, it is another proof of the convergence of the operator to the given function.

4.1 King-type modification of Szász-Mirakjan operators

In 2007, O. Duman and M. Ali Özarslan proposed a King-type modification of Szász-Mirakjan operators[7]. It is defined as

$$\mathcal{S}_n^*(f;x) = e^{-nu_n(x)} \sum_{j=0}^{\infty} \frac{(nu_n(x))^j}{j!} f\left(\frac{j}{n}\right), \tag{4.3}$$

where $x \in [0, \infty)$, $n \in \mathbb{N}$, $f \in \mathcal{C}_b[0, \infty)$ and $u_n(x)$ is a sequence of real valued continuous functions on $[0, \infty)$. If $u_n(x)$ is taken as x, then this opereator becomes the classical Szász-Mirakjan operator. Here, $u_n(x)$ is taken as,

$$u_n(x) = \frac{\sqrt{1 + 4n^2x^2} - 1}{2n}, \quad x \geq 0, \ n \in \mathbb{N}. \tag{4.4}$$

For this operator, its value at the test function is mentioned below which can be checked by simple calculations.

$$\mathcal{S}_n^*(e_0;x) = e_0(x), \quad \mathcal{S}_n^*(e_1;x) = u_n(x) \quad \text{and} \quad \mathcal{S}_n^*(e_2;x) = e_2(x).$$

For the test functions, $\mathcal{S}_n^*(f;x) \to f(x)$. Therefore by the Korovkin-type theorem, $\mathcal{S}_n^*(f) \to f$ for all the continuous functions on $[0, b]$. The following theorem sums up the properties of this modification.

Theorem 4.2. *Let $\mathcal{S}_n^*$ be the operator mentioned in (4.3) and $u_n(x)$ be the sequence of real-valued continuous functions mentioned in (4.4). Then,*

(i) $\mathcal{S}_n^*$ is a positive linear operators on $\mathcal{C}_b[0, \infty)$, $n \in \mathbb{N}$,

(ii) $\mathcal{S}_n^*(e_2;x) = e_2(x) = x^2, \quad n \in \mathbb{N}, x \in [0, \infty)$,

(iii) $\lim_{n\to\infty} \mathcal{S}_n^*(f;x) = f(x)$, on $[0, b]$.

Now, for the estimation of the rate of convergence, the following theorem is to be considered.

Theorem 4.3. *[7] Let $\mathcal{S}_n^*$ be the given operator and $u_n(x)$ metioned in (4.4). Then for $x \in [0, b]$ and for every $f \in \mathcal{C}_b[0, \infty)$,*

$$|\mathcal{S}_n^*(f, x) - f(x)| \leq 2w(f, \delta_x), \tag{4.5}$$

where, $\delta_x = \sqrt{2x(x - u_n(x))}$.

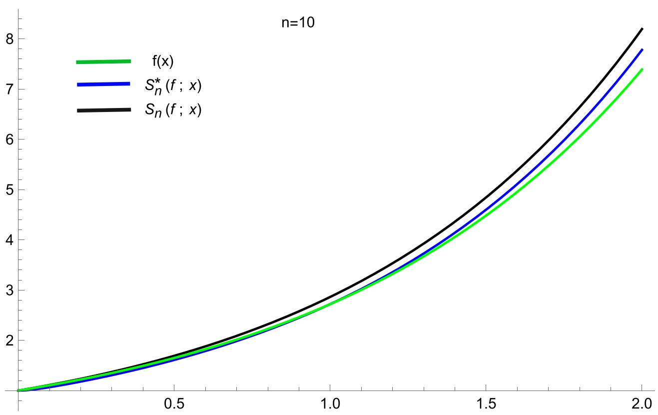
Here, to compare both the estimates of rate of convergence for Szász-Mirakjan and its King-type modifications,
For $x \geq 0$ and $n \in \mathbb{N}$,

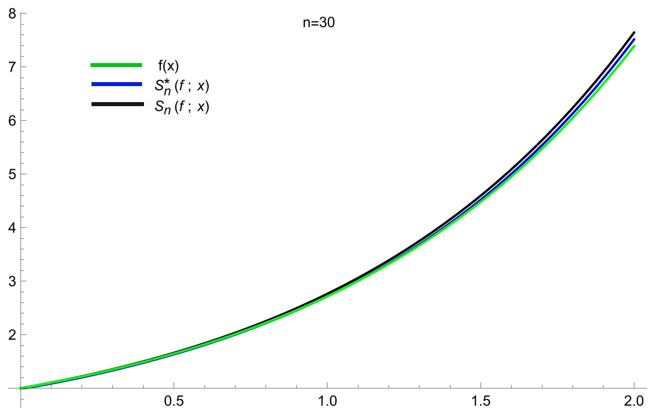
$$\begin{aligned} 4n^2x^2 &\leq 1 + 4n^2x^2 \\ \implies 2nx &\leq \sqrt{1+4n^2x^2} \\ \implies \frac{2nx-1}{2n} &\leq \frac{\sqrt{1+4n^2x^2}-1}{2n} \\ \implies x - \frac{1}{2n} &\leq u_n(x) \\ \implies x - u_n(x) &\leq \frac{1}{2n} \\ \implies \delta_x &\leq \alpha_x. \end{aligned}$$

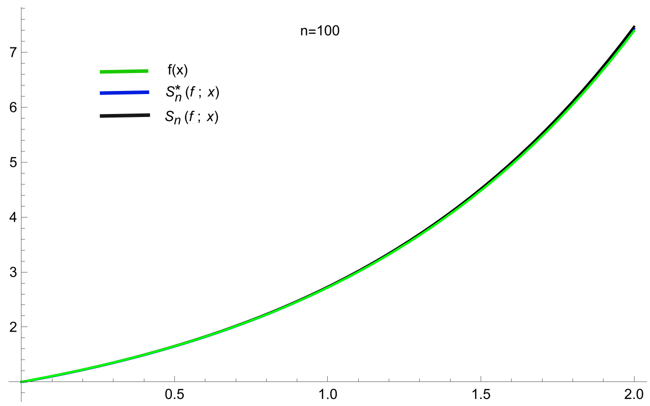
Here, by properties of modulus, it is clear that $w(f, \delta_x) \leq w(f, \alpha_x)$. Therefore, it can be said that estimation of the rate of convergence is better of King type Szász-Mirakjan operators than that of classical Szász-Mirakjan operators.

4.2 Graphical representation for Szász-Mirakjan and its King-type modifications

To show the graphical representation, operators at different values of n are to be considered. Here, graphs are shown for the function, $f(x) = e^x$. Three curves in a graph stand for the function, Szász-Mirakjan operator and King-type modification of the operator.







It can be seen that as n increases, all the curves overlap.

5 Baskakov operator and its King-type modification

The Baskakov operator was given by V.A.Baskakov [8] and it is defined as

$$\mathcal{V}_n(f;x)=\sum_{j=0}^{\infty}\binom{n+j-1}{j}f\left(\frac{j}{n}\right)x^j(1+x)^{-n-j}, \quad n\in\mathbb{N},\ x\in[0,\infty), \tag{5.1}$$

where f belongs to subspace of $[0,\infty)$ for which the above series is convergent. Among these subspaces, space of all bounded and continuous functions on $[0,\infty)$, i.e., $\mathcal{C}_b[0,\infty)$, or the Banach lattice $E[0,\infty)$ given by

$$E[0,\infty)=\left\{f\in\mathcal{C}[0,\infty)\ ;\ \lim_{x\to\infty}\frac{f(x)}{1+x^2}=0\right\}$$

is considered.
It is to be noticed that
$\mathcal{V}_n(e_0;x) = e_0(x), \quad \mathcal{V}_n(e_1;x) = x \quad \text{and} \quad \mathcal{V}_n(e_2;x) = x^2 + \dfrac{x+x^2}{n}.$
Therefore, the korovkin-type theorem satisfies as $\mathcal{V}_n(e_i,x)$ converges to e_i for $i = 0,1,2$. The following theorem gives the estimate for rate of convergence.

Theorem 5.1. *[8] For Baskakov operator defined by* (5) *and* $b > 0$,

$$|\mathcal{V}_n(f;x) - f(x)| \leq w(f;\delta)\left[1 + \frac{1}{\delta}\sqrt{\frac{x+x^2}{n}}\right], \qquad x \in [0,b]. \tag{5.2}$$

It can be seen that as n goes to ∞, error of the approximation goes to zero and hence the operator tends to given function as n tends to ∞.

5.1 King-type modification of Baskakov operators

King-type modification of Baskakov operator was introduced by M. Ali Ozarslan, Oktay Duman and Nazim I. Mahmudov in 2010[9]. It is defined as

$$\mathcal{V}_n^*(f;x) = \sum_{j=0}^{\infty} \binom{n+j-1}{k} f\left(\frac{j}{n}\right)(v_n(x))^j(1+v_n(x))^{-n-j}, \quad n \in \mathbb{N},\ x \in [0,\infty). \tag{5.3}$$

Here, $v_n(x)$ is a sequence of real-valued continuous functions on $[0,\infty)$. If $v_n(x)$ is taken as x, then this operator becomes the classical Baskakov operator. Here, $v_n(x)$ is defined as,

$$v_n(x) = \frac{-1+\sqrt{1+4n(n+1)x^2}}{2(n+1)}, \quad n \in \mathbb{N},\ x \in [0,\infty). \tag{5.4}$$

Values of these operators at test functions are mentioned here,

$$\mathcal{V}_n^*(e_0;x) = e_0(x), \quad \mathcal{V}_n^*(e_1;x) = v_n(x) \quad \text{and} \quad \mathcal{V}_n^*(e_2;x) = e_2(x).$$

Therefore, for the test functions, $\mathcal{V}_n^*(f;x) \to f(x)$. Hence, by the Korovkin theorem, $\mathcal{V}_n^*(f;x)$ converges to $f(x)$ for all continuous functions on $[0,b]$. The following theorem gives a better idea of its properties.

Theorem 5.2. *Let* $\mathcal{V}_n^*(f)$ *be a Baskakov-King operator defined in* (5.4), *where* $\{v_n(x)\}$ *is defined as* (5.4). *Then,*

(i) $\mathcal{V}_n^*$ *is a positive linear operators on* $\mathcal{C}_b[0,\infty)$, $n \in \mathbb{N}$,
(ii) $\mathcal{V}_n^*(e_2;x) = e_2(x) = x^2, \quad n \in \mathbb{N}, x \in [0,\infty)$,
(iii) $\lim_{n\to\infty} \mathcal{V}_n^*(f;x) = f(x)$, *on* $[0,b]$.

Now, for the estimate of error of approximation, following theorem is to be considered.

Theorem 5.3. *[9] For* $\mathcal{V}_n^*(f)$, *a Baskakov-king operator defined in* (5.4) *and* $\{v_n(x)\}$ *is defined as* (5.4), $x \in [0,b]$ *and for every* $x \in \mathcal{C}_b[0,\infty)$

$$|(\mathcal{V}_n^* f)(x) - f(x)| \leq w(f,\delta)\left[1 + \frac{1}{\delta}\sqrt{2x(x - v_n(x))}\right], \quad \delta > 0. \tag{5.5}$$

Now, for the comparison of both estimates of the error of approximation,

$$\begin{aligned} x - v_n(x) &= x - \frac{-1 + \sqrt{1 + 4n(n+1)x^2}}{2(n+1)} \\ &= \frac{2(n+1)x + 1 - \sqrt{1 + 4n(n+1)x^2}}{2(n+1)} \\ &= \frac{1}{2(n+1)} \frac{[2(n+1)x+1]^2 - 1 - 4n(n+1)x^2}{2(n+1)x + 1 + \sqrt{1 + 4n(n+1)x^2}} \\ &= \frac{2(x^2 + x)}{2(n+1)x + 1 + \sqrt{1 + 4n(n+1)x^2}} \\ &\leq \frac{x+1}{2n}. \end{aligned}$$

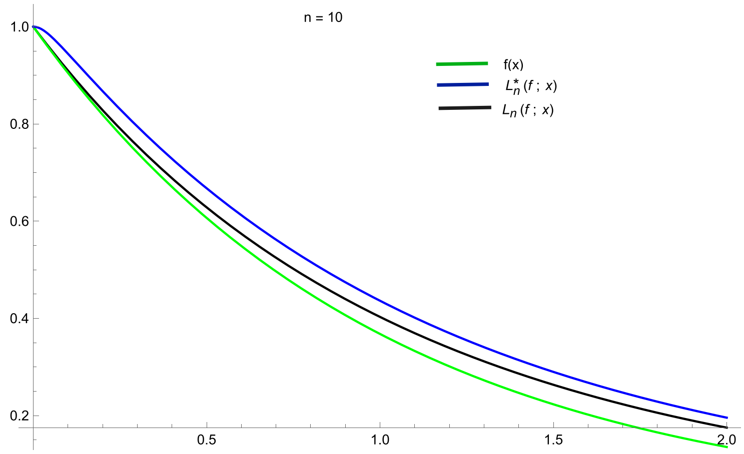
Therefore, we get

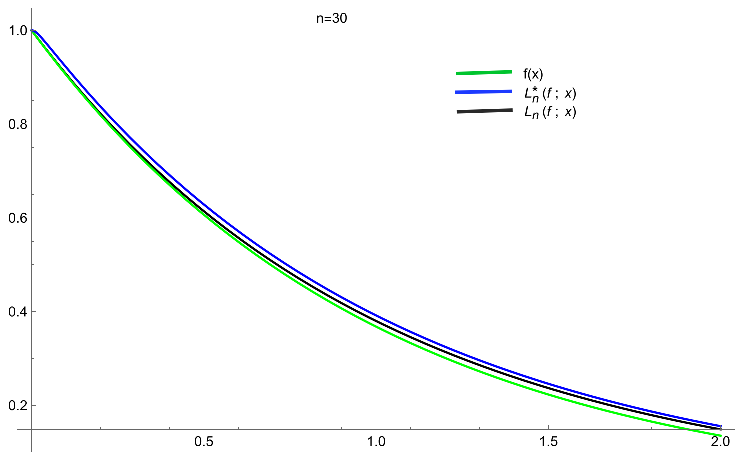
$$2x(x - v_n(x)) \leq \frac{x^2 + x}{n}.$$

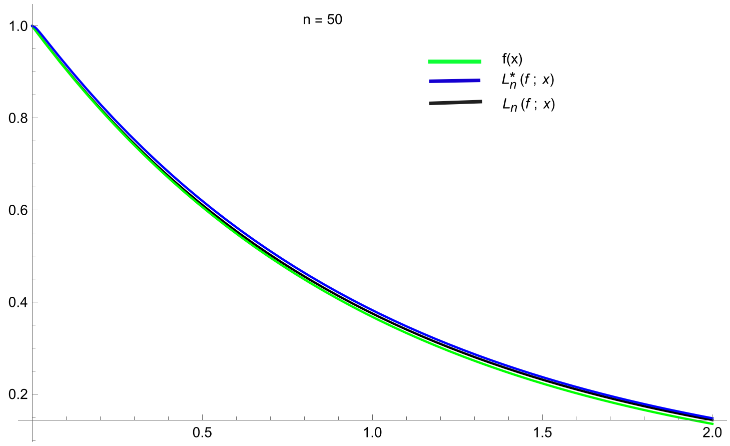
Therefore, error of approximation for King-type Baskakov operators is lesser than the classical Baskakov operators. So, it is a better modification of Baskakov operators.

5.2 Graphical representation for Baskakov and its King-type modifications

To show the graphical representation, operators at different values of n are to be considered. For a function $f(x) = e^{-x}$, a graph of the function, Baskakov operator at that function, and King-type Baskakov operator at the same function are shown for different values of n.







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